

# COMPARISON OF EXPONENTIAL-LOGARITHMIC AND LOGARITHMIC-EXPONENTIAL SERIES

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**ABSTRACT.** We explain how the field of logarithmic-exponential series constructed in [DMM1] and [DMM2] embeds as an exponential field in any field of exponential-logarithmic series constructed in [KK1], [K] and [KS]. On the other hand, we explain why no field of exponential-logarithmic series embeds in the field of logarithmic-exponential series. This clarifies why the two constructions are intrinsically different, in the sense that they produce non-isomorphic models of  $\text{Th}(\mathbb{R}_{\text{an}}, \exp)$ ; the elementary theory of the ordered field of real numbers, with the exponential function and restricted analytic functions.

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## 1. INTRODUCTION

Several constructions of non-archimedean ordered fields endowed with an exponential and a logarithmic map, and with differential operators have appeared in the

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literature in the last two decades. All constructions are based on the use of fields of generalized series (see Definition 2.1). Let us summarize some of the literature highlights. The early works of [D] and [DG] were motivated by Tarski's open problem concerning the decidability of  $T_{\text{exp}}$  := the elementary theory of  $(\mathbb{R}, +, \cdot, <, \exp)$ . In [W], Wilkie established the model completeness of this theory. It became particularly interesting to understand the algebraic structure of the non-archimedean models of  $T_{\text{exp}}$ . In Dahn, [D, p.183], a (non-surjective) exponential map is defined on a field of generalized series. This field, denoted by  $L_0$ , is reconsidered in the works [DMM], [DMM1], [DMM2]. There, the authors construct a non-archimedean model of  $T_{\text{exp}}$ , the so-called field of LE-series. The construction is based on Dahn's field  $L_0$ , and is attributed to Dahn-Göring [DG] in [DMM2, p.63]. The non-standard models constructed in [KK1], [KK2], [KKS], [K], [KS] are based on so-called *prelogarithmic fields* (see Section 3). In [E], [H1], [H2], [KM], and [KM2] the focus is on the differential operators and on power series expansions for solutions of differential equations. Example 4.3 and variants thereof (e.g. with  $\Gamma = \{\log_m x ; m \in \mathbb{N}_0\}$  instead) are considered in [S, 2.3.1].

All constructions described in the literature are complicated and hard to read for the non-specialists. The inherent complexity of the construction makes a comparison of the resulting structures a tour de force. In this paper, we endeavor to explain these different constructions in a unified way. More precisely, we isolate a list of main basic construction steps. We explain the constructions of [DMM1] and [DMM2] and those of [KK1], [K] and [KS] in light of this list. We show how one may combine them in different ways to recover different constructions. This unified approach has several advantages. On the one hand, it makes the constructions more accessible, on the other, it allows to compare the various constructions, up to isomorphism of the obtained models.

The paper is organized as follows: In section 2, we recall some preliminaries on fields of generalized power series. In sections 3, 4, 5 and 6 we explain the construction of exponential-logarithmic power series fields  $\mathbb{R}((G))^{\text{EL}}$  (starting with an arbitrary ordered abelian group  $G$ ; cf. [K]). The construction is split into 3 different steps: the initial data is a given prelogarithmic section (see Section 3 for the definition), which is used to construct in the second step the exponential extension, and finally in the third step the exponential closure. In Section 7, we introduce the notion of morphisms in the category of fields endowed with prelogarithmic sections. Morphisms are used in Section 9 to “run” a generalized construction of logarithmic-exponential fields in exponential-logarithmic fields. To get as a special case an embedding of the LE-series field (cf. [DMM1], [DMM2]) in any EL-series field (cf. [K]) we need to make a special choice of  $G$ . The assumptions on  $G$  are explained and analyzed in Section 8. In Section 10, we then show how to embed the LE-series field of [DMM1], [DMM2] into the exponential closure of the subfield generated by logarithmic words (cf. [H1], [H2], [S]), which is in turn a subfield of any EL-series field. In particular, we get an embedding of Écalle's “trigèbre de transseries”, which is identified as a subfield of the LE-series field in [DMM2, Section 7.1]. The last section 11 is devoted to proving that no non-archimedean

exponential field  $\mathbb{R}((G))^{\text{EL}}$  does embed as an exponential field into the LE-series field.

## 2. REMINDER ON GENERALIZED POWER SERIES.

Throughout,  $\mathbf{k}$  will denote a totally ordered field and  $(G, \cdot, 1, <)$  a multiplicative totally ordered abelian group.

**Definition 2.1.** The field of *generalized power series* over  $G$  with coefficients in  $\mathbf{k}$  is defined as follows:

$$\mathbf{k}((G)) := \left\{ \alpha = \sum \alpha(g) g \mid \alpha(g) \in \mathbf{k}, \text{ supp } \alpha \text{ is anti well ordered} \right\}.$$

Here  $\text{supp } \alpha := \{g \in G \mid \alpha(g) \neq 0\}$  is the support of  $\alpha$ . Addition and multiplication of series are defined in the usual manner. See [N] for more details.

The field  $\mathbf{k}((G))$  is endowed with a *canonical valuation*  $v : \mathbf{k}((G)) \rightarrow G \cup \{0\}$ , defined by  $v(0) = 0$  where  $0 < G$ ,  $v(\alpha) := \max \text{supp } \alpha$  for  $\alpha \neq 0$ . The valuation verifies that  $v(\alpha\alpha') = v(\alpha) \cdot v(\alpha')$ , as well as the ultrametric triangle inequality:  $v(\alpha + \alpha') \leq \max\{v(\alpha), v(\alpha')\}$ , with equality if  $v(\alpha) \neq v(\alpha')$ . Thus  $(\mathbf{k}((G)), v)$  is a *valued field* with *value group*  $G$  ( $G$  is also called the group of exponents, or the group of monomials), and *residue field*  $\mathbf{k}$ . The field  $\mathbf{k}((G))$  comes equipped with the *anti lexicographic order*:  $0 \neq \alpha \in \mathbf{k}((G))$  is  $> 0$  if  $\alpha(v(\alpha)) > 0$ . The valuation  $v$  is *compatible* with this order, i.e. for all  $\alpha_1, \alpha_2$  with  $0 < \alpha_1 < \alpha_2$  we have  $v(\alpha_1) \leq v(\alpha_2)$ .

We identify  $\mathbf{k}$  with the ordered subfield  $\{\alpha \cdot 1 \mid \alpha \in \mathbf{k}\}$  of  $\mathbf{k}((G))$ . Similarly, we identify  $G$  with the ordered subgroup  $\{1 \cdot g \mid g \in G\}$  of  $\mathbf{k}((G))^\times$ . Hence  $G > 0$  in  $\mathbf{k}((G))$  and one might think of  $G$  as a multiplicative group of germs of functions  $\mathbf{k} \rightarrow \mathbf{k}$  at  $+\infty$ .

For any  $g \in G$  we denote by  $G^{<g}$  the strict initial segment of  $G$  determined by  $g$ , similar notations for  $G^{\leq g}$ ,  $G^{>g}$ , and  $G^{\geq g}$ . More generally, for any subset  $A \subseteq G$  we denote by  $G^{<A} := \{g \mid g < a, \text{ for all } a \in A\}$ , similar notations for  $G^{\leq A}$ ,  $G^{>A}$ ,  $G^{\geq A}$ . For any subset  $S \subseteq G$ , let  $\mathbf{k}((S)) := \{\alpha \in \mathbf{k}((G)) \mid \text{supp } \alpha \subseteq S\}$ . Since  $\text{supp}(\alpha_1 + \alpha_2) \subseteq \text{supp } \alpha_1 \cup \text{supp } \alpha_2$  and  $\text{supp}(-\alpha) = \text{supp } \alpha$ ,  $(\mathbf{k}((S)), +)$  is a  $\mathbf{k}$ -linear subspace of  $(\mathbf{k}((G)), +)$ .

The *valuation ring*  $\mathbf{k}((G^{\leq 1})) = \mathbf{k} \oplus \mathbf{k}((G^{<1}))$  is a convex subring of  $\mathbf{k}((G))$  with maximal ideal  $\mathbf{k}((G^{<1}))$ . If  $\alpha \in \mathbf{k}((G))$ ,  $\alpha > 0$  and  $g := \max \text{supp } \alpha$ , then  $g^{-1} \cdot \alpha \in \mathbf{k}((G^{\leq 1}))$ , hence there are uniquely determined  $a \in \mathbf{k}^{>0}$ ,  $\varepsilon \in \mathbf{k}((G^{<1}))$  such that  $\alpha = g \cdot a \cdot (1 + \varepsilon)$ . We have the following direct sum (respectively, multiplicative direct sum) decompositions:

$$(1) \quad \mathbf{k}((G)) = \mathbf{k}((G^{>1})) \oplus \mathbf{k} \oplus \mathbf{k}((G^{<1})),$$

$$(2) \quad \mathbf{k}((G))^{>0} = G \cdot \mathbf{k}^{>0} \cdot (1 + \mathbf{k}((G^{<1}))).$$

**Remark 2.2.** In the literature, and in particular in the papers of the first author cited here, the group  $G$  is written additively,  $g \in G$  is associated to the monomial  $t^g$  and series in  $\mathbf{k}((G))$  are written  $\sum \alpha_g t^g$  and have well ordered support. The map

$g \mapsto t^g$  is an order *reversing* isomorphism, the order on  $\mathbf{k}((G))$  is the lexicographic order, and the canonical valuation is defined by  $v(\alpha) := \min \text{supp } \alpha$ . In the present paper, we have opted for the multiplicative notation because it is more suitable for functional interpretations (cf. [K2] and [KM]).

### 3. PRELOGARITHMIC SECTIONS AND ASSOCIATED PRELOGARITHMS.

In this and the following three sections, we shall recall the construction, and give examples of exponential-logarithmic series fields. We begin by recalling the definition of the restricted logarithm defined on the group of 1-units of the valuation ring of  $\mathbf{k}((G))$ .

**Definition 3.1.** The *logarithm on 1-units* is the map:

$$(3) \quad 1 + \mathbf{k}((G^{<1})) \longrightarrow \mathbf{k}((G^{<1})) \text{ defined by } 1 + \varepsilon \mapsto \sum_{i \geq 1} (-1)^{(i-1)} \frac{\varepsilon^i}{i}.$$

This map is an isomorphism of ordered groups (cf. [A]).

We now focus on extending the domain of the logarithm to  $\mathbf{k}((G))^{>0}$ .

**Definition 3.2.** A *prelogarithmic section*  $l$  of  $\mathbf{k}((G))$  is an embedding of ordered groups  $l : (G, \cdot) \longrightarrow (\mathbf{k}((G^{>1})), +)$ . The tuple  $(\mathbf{k}((G)), l)$  is called a *series field with prelogarithmic section*  $l$ ; if  $l$  is clear from the context we just say  $\mathbf{k}((G))$  is a *prelogarithmic series field*.

Note that prelogarithmic sections on  $\mathbf{k}((G))$  exist, cf. Remark 4.2 and Example 4.3 below.

**Definition 3.3.** Let  $\mathbf{k}$  be an ordered field and  $\log : (\mathbf{k}^{>0}, \cdot) \longrightarrow (\mathbf{k}, +)$  an order preserving embedding of groups. We say that  $(\mathbf{k}, \log)$  is an *ordered prelogarithmic field*. If  $\log$  is surjective we say that  $(\mathbf{k}, \log)$  is an *ordered logarithmic field*, or that  $(\mathbf{k}, \exp)$  is an *ordered exponential field*, where  $\exp := \log^{-1}$ .

Let  $(\mathbf{k}, \log)$  be an ordered prelogarithmic field and  $G$  be a multiplicative, abelian ordered group. Let  $l$  be a prelogarithmic section of  $\mathbf{k}((G))$ . If  $\alpha \in \mathbf{k}((G))$ ,  $\alpha > 0$  and  $g := \max \text{supp } \alpha$ , write  $\alpha = g \cdot a \cdot (1 + \varepsilon)$  with  $a \in \mathbf{k}^{>0}$ ,  $\varepsilon \in \mathbf{k}((G^{<1}))$ .

**Definition 3.4.** Define

$$(4) \quad L(\alpha) = L(g \cdot a \cdot (1 + \varepsilon)) = l(g) + \log a + \sum_{i \geq 1} (-1)^{(i-1)} \frac{\varepsilon^i}{i}$$

Then  $L : (\mathbf{k}((G))^{>0}, \cdot) \longrightarrow (\mathbf{k}((G)), +)$  is the uniquely determined order preserving embedding of groups, extending  $\log$ ,  $l$  and the logarithm on 1-units. We call  $L$  the *prelogarithm* associated to the prelogarithmic section  $l$  on  $\mathbf{k}((G))$ .

## 4. MAIN EXAMPLES OF PRELOGARITHMIC SERIES FIELDS

**Definition 4.1.** Let  $\Gamma$  be a totally ordered set, and  $\mathbf{k}$  a totally ordered field. We shall begin by defining  $\Gamma^{\mathbf{k}}$ ; the *Hahn group* of rank  $\Gamma$  with exponents in  $\mathbf{k}$ , written multiplicatively. Consider a totally ordered set of variables

$$X := \{ x_\gamma \mid \gamma \in \Gamma \}$$

and let  $\Gamma^{\mathbf{k}}$  be the set of formal products of the form

$$g = \prod_{\gamma \in \Gamma} x_\gamma^{g(\gamma)}$$

where  $g(\gamma) \in \mathbf{k}$  and  $\text{supp } g := \{ \gamma \in \Gamma \mid g(\gamma) \neq 0 \}$  is anti wellordered in  $\Gamma$ . Multiplication is defined pointwise:

$$g_1 g_2 = \prod_{\gamma \in \Gamma} x_\gamma^{g_1(\gamma) + g_2(\gamma)}$$

so 1 is the product with empty support. The order is anti lexicographic:  $g > 1$  if  $g(\gamma) > 0$  where  $\gamma = \max \text{supp } g$ . For example  $x_\gamma > 1$  for all  $\gamma \in \Gamma$ . Thus,  $\Gamma^{\mathbf{k}}$  is a totally ordered abelian group.

**Remark 4.2.** Hahn's Embedding Theorem [H] states that every totally ordered abelian group  $G$  is (isomorphic to) a subgroup of a Hahn group  $\Gamma^{\mathbb{R}}$ , for a suitably chosen  $\Gamma$ . Therefore when constructing prelogarithmic series fields  $\mathbb{R}((G))$  one may assume without loss of generality that  $G$  is a subgroup of a Hahn group.

**Example 4.3.** Set  $G = \Gamma^{\mathbf{k}}$ , we define the *basic prelogarithmic section* on  $\mathbf{k}((G))$  by :

$$l\left(\prod_{\gamma \in \Gamma} x_\gamma^{g(\gamma)}\right) = \sum_{\gamma \in \Gamma} g(\gamma) x_\gamma.$$

Now assume as above that  $\mathbf{k}$  admits a  $\log : (\mathbf{k}^{>0}, \cdot) \rightarrow (\mathbf{k}, +)$ , and let  $L$  be the associated *basic prelogarithm* as given by (4). We denote by  $\mathbf{k}((\Gamma^{\mathbf{k}}))^L$  the prelogarithmic series field thus constructed.

Our aim is to use prelogarithmic series fields to construct ordered exponential fields  $(K, E)$  which are models of  $T_{\text{exp}}$ . So we are mainly interested in exponentials satisfying the *growth axiom* scheme:

$$(\mathbf{GA}) \quad \alpha \in K, \alpha \geq n^2 \implies E(\alpha) > \alpha^n \quad (n \geq 1)$$

Note that because of the hypothesis  $\alpha \geq n^2$ , **(GA)** is only relevant for  $v(\alpha) \geq 1$ . Let us consider the case  $v(\alpha) > 1$ . In this case, " $\alpha > n^2$ " holds for all  $n \in \mathbb{N}$  if  $\alpha$  is positive, and the axiom scheme **(GA)** is thus equivalent to the assertion

$$(5) \quad \forall n \in \mathbb{N} : E(\alpha) > \alpha^n.$$

Applying  $L := E^{-1}$  on both sides, we find that this is equivalent to

$$(6) \quad \forall n \in \mathbb{N} : \alpha > L(\alpha^n) = nL(\alpha).$$

Via the valuation  $v$ , this in turn holds if

$$(7) \quad v(L(\alpha)) < v(\alpha) .$$

**Definition 4.4.** A (pre)logarithm  $L$  on the ordered field  $K$  will be called a **(GA)**-(pre)logarithm if it satisfies (7) for  $\alpha \in K^{>0}$  with  $v(\alpha) > 1$ .

**Example 4.5.** The basic prelogarithm  $L$  on  $\mathbf{k}((G))$  (given in Example 4.3) does *not* satisfy **(GA)** (e. g.  $L(x_\gamma) = x_\gamma$ ). To remedy to this problem, we fix an embedding

$$\sigma : \Gamma \longrightarrow \Gamma$$

which is decreasing (i.e.  $\sigma(\gamma) < \gamma$  for all  $\gamma \in \Gamma$ ), and define the induced prelogarithmic section  $l_\sigma$  as follows:

$$l_\sigma\left(\prod_{\gamma \in \Gamma} x_\gamma^{g(\gamma)}\right) = \sum_{\gamma \in \Gamma} g(\gamma) x_{\sigma(\gamma)} .$$

The associated prelogarithm ( given in (4) ) is denoted by  $L_\sigma$ . We note that  $L_\sigma$  satisfies **(GA)**: one verifies that (7) holds for  $\alpha \in \mathbf{k}((G))^{>0}$ , with  $g := v(\alpha) > 1$ . Set  $x_\gamma := \max \text{supp } g$ ; one verifies that

$$x_{\sigma(\gamma)} = v(l_\sigma(g)) = v(L_\sigma(\alpha)) < v(\alpha) = g \text{ in } G^{>1} .$$

We denote by  $\mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma L}$  the prelogarithmic series field thus constructed.

**Example 4.6.** In Example 4.3 above,  $\Gamma$  can consist of a totally ordered set of germs at infinity of non-oscillating real valued functions of a real variable, which are infinitely large and positive (i.e.  $\lim_{x \rightarrow +\infty} f(x) = +\infty$ ). For example take

$$\Gamma := \{\log_m x \mid m \in \mathbb{Z}\} .$$

Here,  $\log$  is the natural logarithm on  $\mathbb{R}$  and  $\log_m$  is its  $m$ -th iterate (for  $n \in \mathbb{N}$ ,  $\log_{-n} = \exp_n$  the  $n$ -th iterate of the exponential). Let

$$\sigma : \log_m x \mapsto \log_{m+1} x .$$

Let  $G$  be the group of *exponential-logarithmic words*; that is, the subgroup of  $\Gamma^{\mathbb{R}}$  consisting of products with *finite* support. As in Example 4.3 above, the induced prelogarithmic section  $l_\sigma$  on  $\mathbb{R}((G))$  is:

$$l_\sigma(x^{r_0}(\log x)^{r_1} \cdots (\log_n x)^{r_n}) = r_0 \log x + \cdots + r_n \log_{n+1} x .$$

## 5. THE EXPONENTIAL EXTENSION OF A PRELOGARITHMIC FIELD.

The following construction or close variants of it, namely to add exponentials to a given field with prelogarithmic section, is used by [D], [E], [DMM1], [DMM2], [H1], [H2], [KK1], [KK2], [K], [KS] and [S]. It is the core step in constructing exponentials of infinitely large elements.

From now on, we fix a prelogarithmic series field  $(\mathbf{k}((G)), l)$ . Observe that  $l : G \longrightarrow (\mathbf{k}((G^{>1})), +)$  is not surjective if  $G \neq \{1\}$  (cf. [KKS]). More precisely, the exponential of any element in  $\mathbf{k}((G^{>1})) \setminus l(G)$  is not defined. We shall enlarge our group of monomials  $G$  to a group extension  $G^\#$  to include the missing exponentials.

We take  $G^\#$  to be a *multiplicative* copy  $e[\mathbf{k}((G^{>1}))]$  of  $\mathbf{k}((G^{>1}))$  over  $l(G)$ . More precisely, we construct  $G^\#$  formally as follows:

$$G^\# := \{e(\alpha) \mid \alpha \in \mathbf{k}((G^{>1}))\}, \text{ where } e(\alpha) := g \text{ if } \exists g \in G \text{ s.t. } \alpha = l(g) \}$$

By its definition,  $G$  is a subset of  $G^\#$ . We define multiplication on  $G^\#$  as follows:  $e(\alpha_1)e(\alpha_2) := e(\alpha_1 + \alpha_2)$ . In particular, if  $g_1 = e(\alpha_1)$ ,  $g_2 = e(\alpha_2) \in G$ , then  $e(\alpha_1)e(\alpha_2) = e(l(g_1) + l(g_2)) = e(l(g_1g_2)) = g_1g_2$ , so  $G$  is a *subgroup* of  $G^\#$ . We equip  $G^\#$  with a total order:  $e(\alpha_1) < e(\alpha_2)$  if and only if  $\alpha_1 < \alpha_2$  in  $\mathbf{k}((G^{>1}))$ . Again, if  $g_1 = e(\alpha_1)$ ,  $g_2 = e(\alpha_2) \in G$ , then  $e(\alpha_1) < e(\alpha_2)$  if and only if  $l(g_1) < l(g_2)$  in  $\mathbf{k}((G^{>1}))$  if and only if  $g_1 < g_2$  in  $G$ , so  $G$  is an *ordered subgroup* of  $G^\#$ . Since  $G \subseteq G^\#$  as ordered abelian multiplicative groups, we view  $\mathbf{k}((G))$  as an ordered subfield of  $\mathbf{k}((G^\#))$  by identifying  $\mathbf{k}((G))$  with the elements of  $\mathbf{k}((G^\#))$  having support in  $G$ .

One verifies that the map  $l^\# : (G^\#, \cdot) \longrightarrow (\mathbf{k}((G^\#)^{>1}), +)$  defined by  $l^\#(e(\alpha)) := \alpha$  for  $\alpha \in \mathbf{k}((G^{>1}))$  is a prelogarithmic section with  $l^\#(G^\#) = \mathbf{k}((G^{>1}))$  and  $l^\#$  extends  $l$  on  $G$ . By construction of the logarithms  $L$  and  $L^\#$  on  $\mathbf{k}((G))^{>0}$  and  $\mathbf{k}((G^\#))^{>0}$  respectively,  $L^\#$  is an extension of  $L$ .

**Definition 5.1.** We define the *exponential extension* of  $(\mathbf{k}((G)), L)$  to be  $(\mathbf{k}((G^\#)), L^\#)$ .

We have the following commutative diagram of embeddings (whenever we have a set theoretic inclusion we use the arrow  $\hookrightarrow$ ):

$$(8) \quad \begin{array}{ccccc} & & \mathbf{k}((G^\#))^{>0} & \xrightarrow{L^\#} & \mathbf{k}((G^\#)) \\ & \uparrow & & \nearrow & \uparrow \\ G^\# & \xrightarrow{l^\#} & \mathbf{k}((G^\#)^{>1}) & & \\ & \searrow \cong & \uparrow & & \\ G & \xrightarrow{l} & \mathbf{k}((G^{>1})) & \hookrightarrow & \mathbf{k}((G)) \end{array}$$

## 6. THE EXPONENTIAL CLOSURE OF A PRELOGARITHMIC FIELD

From now on we assume that the given logarithm on the residue field  $\log : (\mathbf{k}^{>0}, \cdot) \longrightarrow (\mathbf{k}, +)$  is surjective.

**Definition 6.1.** If  $n = 0$  set  $(\mathbf{k}((G))^{\#n}, L^{\#n}) := (\mathbf{k}((G)), L)$ . For  $n \in \mathbb{N}$ , define inductively the  $n$ -th *exponential extension* of  $(\mathbf{k}((G)), L)$ :  $(\mathbf{k}((G))^{\#n}, L^{\#n}) :=$  the exponential extension of  $(\mathbf{k}((G^{\#n-1})), L^{\#n-1})$ .

Hence  $\mathbf{k}((G))^{\#n} = \mathbf{k}((G^{\#n}))$ . Here, the prelogarithm on  $\mathbf{k}((G^{\#n}))^{>0}$  associated to  $l^{\#n}$  is denoted by  $L^{\#n}$ .

**Definition 6.2.** Set  $\mathbf{k}((G))^{\text{EL}} := \bigcup_{n \in \mathbb{N}_0} \mathbf{k}((G))^{\#n}$  and  $\text{Log} := \bigcup_{n \in \mathbb{N}_0} L^{\#n}$ . We call  $(\mathbf{k}((G))^{\text{EL}}, \text{Log})$  is the EL-series field over the prelogarithmic field  $(\mathbf{k}((G))l)$ .

Below, we gather some properties of  $(\mathbf{k}((G))^{\text{EL}}, \text{Log})$ . We note that  $\mathbf{k}((G))$  is contained in the image of  $L^{\#}$ : the image of  $L^{\#}$  contains  $\mathbf{k}((G^{<1}))$  (from the logarithm on 1-units),  $\mathbf{k}((G^{>1}))$  (as image of  $l^{\#}$ ) and  $\mathbf{k}$ , since  $\log$  is surjective. By induction we have

$$(9) \quad \mathbf{k}((G^{\#n})) \subseteq L^{\#n+1}[\mathbf{k}((G^{\#n+1}))^{>0}]$$

and

$$(10) \quad \mathbf{k}((G^{\#n}, >^1)) = L^{\#n+1}(G^{\#n+1}).$$

So  $\text{Log} : (\mathbf{k}((G))^{\text{EL}}, >^0, \cdot) \rightarrow (\mathbf{k}((G))^{\text{EL}}, +)$  is an order preserving *isomorphism*. Let  $\text{Exp} : (\mathbf{k}((G))^{\text{EL}}, +) \rightarrow (\mathbf{k}((G))^{\text{EL}}, >^0, \cdot)$  denote the inverse of  $\text{Log}$ . By (9) and (10), we have

$$(11) \quad \text{Exp}[\mathbf{k}((G^{\#n}))] \subseteq \mathbf{k}((G^{\#n+1}))^{>0}$$

and

$$(12) \quad \text{Exp}[\mathbf{k}((G^{\#n}, >^1))] = G^{\#n+1}.$$

Finally, although we do not use this fact, we note that  $\mathbf{k}((G))^{\text{EL}} \subseteq \mathbf{k}((G^{\text{EL}}))$  where  $G^{\text{EL}} := \bigcup_n G^{\#n}$ .

**Example 6.3.** We consider the prelogarithmic field  $\mathbf{k}((\Gamma^{\mathbf{k}}))^L$  constructed in Example 4.3. The EL series field obtained as above by forming its exponential closure shall be henceforth denoted by  $\mathbf{k}((\Gamma^{\mathbf{k}}))^{\text{EL}}$ .

**Example 6.4.** We consider the prelogarithmic field  $\mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma L}$  constructed in Example 4.5. The EL series field obtained as above by forming its exponential closure shall be henceforth denoted by  $\mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma \text{EL}}$ . We claim that  $\text{Log}_{\sigma} : (\mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma \text{EL}}, >^0, \cdot) \rightarrow (\mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma \text{EL}}, +)$  satisfies **(GA)**. In Example 4.5, we have shown that

$$(13) \quad v(l_{\sigma}(g)) < g \text{ for } g \in G^{>1}.$$

It suffices to show that this property is preserved by exponential extension. Let  $g^{\#} \in G^{\#, >^1}$ . Set  $g := v(l_{\sigma}^{\#}(g^{\#})) \in G^{>1}$ . Applying (13) we get:

$$v(l_{\sigma}(v(l_{\sigma}^{\#}(g^{\#})))) < v(l_{\sigma}^{\#}(g^{\#})).$$

Since  $l_{\sigma}(v(l_{\sigma}^{\#}(g^{\#}))) > 0$  and  $l_{\sigma}^{\#}(g^{\#}) > 0$ , we get that (by compatibility of  $v$  with the order):

$$l_{\sigma}(v(l_{\sigma}^{\#}(g^{\#}))) < l_{\sigma}^{\#}(g^{\#}).$$

Since  $l_{\sigma}^{\#}$  extends  $l_{\sigma}$ , the last inequality reads

$$l_{\sigma}^{\#}(v(l_{\sigma}^{\#}(g^{\#}))) < l_{\sigma}^{\#}(g^{\#}).$$

Since  $l_{\sigma}^{\#}$  is order preserving, we conclude that  $v(l_{\sigma}^{\#}(g^{\#})) < g^{\#}$ , as required.



## 7. MORPHISMS OF PRELOGARITHMIC FIELDS.

In this section, we define a morphism on a prelogarithmic series field  $(\mathbf{k}((G)), l)$  induced by a morphism  $\psi$  of  $G$ , and use it in turn to induce a special automorphism (“value group induced” automorphism)  $\psi^{EL}$  of the corresponding EL-series field  $(\mathbf{k}((G))^{EL}, \text{Log})$ . These automorphisms  $\psi^{EL}$  will play the role of “Log-substitution maps” (cf. Example 7.6); symbolically  $\psi^{EL}(f(x)) = f(\text{Log}(x))$  (substituting  $\text{Log}(x)$  for  $x$ ). The aim is to recover in Section 10 the inverse map  $(\psi^{EL})^{-1}$  (substituting  $\text{Exp}(x)$  for  $x$ ) which plays a key role in the construction of the LE series field  $\mathbf{k}((x^{-1}))^{LE}$  of [DMM2].

**Definition 7.1.** Let  $\nu : \mathbf{k}((G_1)) \rightarrow \mathbf{k}((G_2))$  be a homomorphism of  $\mathbf{k}$ -vector spaces. We say that  $\nu$  *respects arbitrary sums* if it commutes with arbitrary sums, i. e. if  $\nu(\sum a_g g) = \sum a_g \nu(g)$ .

Let  $(\mathbf{k}((G_1)), l_1)$  and  $(\mathbf{k}((G_2)), l_2)$  be series fields with prelogarithmic sections and let  $\psi : G_1 \rightarrow G_2$  be an order preserving embedding (of groups). Then  $\psi$  extends to  $\mathbf{k}((G_1))$  in the obvious way and the resulting extension is also denoted by  $\psi$ . Since  $\psi$  respects arbitrary sums (from the way it is defined),  $\psi$  commutes with the logarithms on 1-units defined in (3).

**Definition 7.2.** We call  $\psi$  a *morphism* from  $(\mathbf{k}((G_1)), l_1)$  to  $(\mathbf{k}((G_2)), l_2)$  if  $\psi$  commutes with the prelogarithmic sections, that is:  $\psi \circ l_1 = l_2 \circ \psi$  on  $G_1$ :

$$(14) \quad \begin{array}{ccc} \mathbf{k}((G_1)) & \xrightarrow{\psi} & \mathbf{k}((G_2)) \\ \uparrow & & \uparrow \\ \mathbf{k}((G_1^{>1})) & \xrightarrow{\quad} & \mathbf{k}((G_2^{>1})) \\ \uparrow & & \uparrow \\ G_1 & \xrightarrow{\psi} & G_2 \end{array}$$

$l_1$   $l_2$

If this is the case, then  $\psi$  also commutes with the prelogarithms:

$$\psi \circ L_1 = L_2 \circ \psi \text{ on } \mathbf{k}((G_1))^{>0}.$$

(Indeed, if  $a \in \mathbf{k}((G_1))^{>0}$  write  $a = g \cdot r \cdot (1 + \varepsilon)$  and compute:

$$\psi(L_1(a)) = \psi(l_1 g + \log r + \log(1 + \varepsilon)) = \psi l_1(g) + \log r + \psi(\log(1 + \varepsilon)) = l_2 \psi(g) + \log r + \log(1 + \psi \varepsilon) = L_2(\psi(a)).$$

If  $\psi$  is such a morphism, then  $\psi$  induces a morphism

$$\psi^\# : (\mathbf{k}((G_1^\#)), l_1^\#) \rightarrow (\mathbf{k}((G_2^\#)), l_2^\#)$$

defined as follows: for  $g^\# \in G_1^\#$

$$\psi^\#(g^\#) = (l_2^\#)^{-1} \circ \psi \circ l_1^\#(g^\#).$$

Observe that  $\psi \circ l_1^\#(g^\#) \in \mathbf{k}((G_2^{>1}))$  is indeed in the image of  $l_2^\#$ , and that  $\psi^\#$  extends  $\psi$ .

Here is the situation in a diagram. The part of the diagram with non broken arrows, commutes:

(15)

Adding the prelogarithmic sections of diagram 8 to diagram 15 we get a complete illustration of the constructions so far in the following figure:

(16)

We now iterate the  $\sharp$ -construction: by induction we get a morphism

$$\psi^{\sharp n} : (\mathbf{k}((G_1^{\sharp n})), l_1^{\sharp n}) \longrightarrow (\mathbf{k}((G_2^{\sharp n})), l_2^{\sharp n})$$

extending  $\psi^{\#n-1}$ . In the following two lemmas, we record for later use properties of the resulting map

$$\psi^{EL} : \mathbf{k}((G_1))^{EL} \longrightarrow \mathbf{k}((G_2))^{EL},$$

defined by

$$\psi^{EL} := \bigcup_{n \in \mathbb{N}} \psi^{\#n}.$$

**Proposition 7.3.** *The following properties hold.*

- (1)  $\psi^{EL}$  is a  $\mathbf{k}$ -embedding of ordered fields which respects arbitrary sums.
- (2)  $\psi^{EL}$  respects Log and Exp: for  $a \in \mathbf{k}((G_1))^{EL}$ ;  $\psi^{EL}(\text{Log}(a)) = \text{Log}(\psi^{EL}(a))$  (if  $a > 0$ ) and  $\psi^{EL}(\text{Exp}(a)) = \text{Exp}(\psi^{EL}(a))$ .
- (3) If  $G_2 \subseteq \psi^{EL}(G_1^{\#n})$  for some  $n \in \mathbb{N}$  then  $\mathbf{k}((G_2^{\#m}))$  is in the image of  $\mathbf{k}((G_1^{\#n+m}))$  under  $\psi^{EL}$  for all  $m \in \mathbb{N}$ . Hence in this case  $\psi^{EL}$  is surjective and  $(\psi^{EL})^{-1}$  respects arbitrary sums.

*Proof.* (1) and (2) are inherited from the corresponding properties of  $\psi^{\#}$ . To see (3), note that by definition,  $\mathbf{k}((G_2))$  is in the image of  $\psi^{EL}|_{\mathbf{k}((G_1^{\#n}))}$ . Thus  $\mathbf{k}((G_2^{\#m}))$  is in the image of  $\psi^{EL}|_{\mathbf{k}((G_1^{\#n+m}))}$  for all  $m \in \mathbb{N}$ . ■

**Definition 7.4.** Assume that  $G = G_1 = G_2$  and that  $l = l_1 = l_2$ . We say that the morphism  $\psi : G \longrightarrow G$  is a *contracting* morphism if for all  $g \in G^{>1}$   $\psi(g) < g$ . Similarly, we say that  $\psi^{EL}$  is a *contraction* if  $\psi^{EL}(g) < g$  for all  $g \in G^{EL}$ ,  $g > 1$ .

Contracting morphisms will play a key role in section 10. We record the following for later use.

**Proposition 7.5.** *Assume that  $G = G_1 = G_2$  and that  $l = l_1 = l_2$ , and that  $\psi$  is a contracting morphism, then  $\psi^{EL}$  is a contraction*

*Proof.* By induction, it is enough to show  $\psi^{\#}(g^{\#}) < g^{\#}$  for all  $g^{\#} \in G^{\#}$ ,  $g^{\#} > 1$ . Since  $\psi(g) < g$  ( $g \in G$ ,  $g > 1$ ) we have  $\psi(f) < f$  for all  $f \in \mathbf{k}((G^{>1}))$ ,  $f > 0$ . Take  $g^{\#} \in G^{\#}$ ,  $g^{\#} > 1$ . Then  $l^{\#}(g^{\#}) \in \mathbf{k}((G^{>1}))$  is positive, Thus  $\psi(l^{\#}(g^{\#})) < l^{\#}(g^{\#})$  and since  $(l^{\#})^{-1}$  is monotone, we get  $\psi^{\#}(g^{\#}) = (l^{\#})^{-1} \circ \psi \circ l^{\#}(g^{\#}) < (l^{\#})^{-1}(l^{\#}(g^{\#})) = g^{\#}$  as desired. ■

**Example 7.6.** We study the morphism induced by  $\sigma$  on the exponential-logarithmic series field  $\mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma EL}$  examined in Examples 4.5 and 6.4. The embedding  $\sigma : \Gamma \longrightarrow \Gamma$  lifts to an order preserving embedding  $\psi_{\sigma} := \tilde{\sigma} : G \longrightarrow G$  of  $G := \Gamma^{\mathbf{k}}$  into itself, defined by

$$\psi_{\sigma}\left(\prod_{\gamma \in \Gamma} x_{\gamma}^{g(\gamma)}\right) = \prod_{\gamma \in \Gamma} x_{\sigma(\gamma)}^{g(\gamma)} = \prod_{\gamma \in \Gamma} l_{\sigma}(x_{\gamma})^{g(\gamma)}.$$

It is straightforward to verify that  $\psi_{\sigma}$  is a morphism, which extends to  $\psi^{\sigma EL} : \mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma EL} \longrightarrow \mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma EL}$ . Note also that for all  $g \in G^{>1}$   $\psi_{\sigma}(g) < g$ , so that  $\psi^{\sigma EL}(g) < g$  for all  $g \in G^{EL}$ ,  $g > 1$  (by Proposition 7.5). We also apply Proposition 7.3 (3) to establish that  $\psi^{\sigma EL}$  is surjective (even if  $\sigma$  is not): it suffices to show that for all  $g \in G$  we have that  $g \in \psi_{\sigma}^{\#}(G^{\#})$ . Let  $g = \prod_{\gamma \in \Gamma} x_{\gamma}^{g(\gamma)} \in G$ . Now  $\sum_{\gamma \in \Gamma} g(\gamma) x_{\gamma} \in \mathbf{k}((G^{>1})) = l^{\#}(G^{\#})$ . Let  $g^{\#} \in G^{\#}$  be

such that  $l_\sigma^\#(g^\#) = \sum_{\gamma \in \Gamma} g(\gamma) x_\gamma$ . We compute:  $\psi_\sigma^\#(g^\#) = l_\sigma^{\#-1} \circ \psi_\sigma \circ l_\sigma^\#(g^\#) = l_\sigma^{\#-1} \circ \psi_\sigma(\sum_{\gamma \in \Gamma} g(\gamma) x_\gamma) = l_\sigma^{\#-1}(\sum_{\gamma \in \Gamma} g(\gamma) x_{\sigma(\gamma)}) = g$  as required.

## 8. GROUPS SATISFYING THE GROWTH AXIOM.

**Lemma 8.1.** *Let  $A, B$  be subgroups of the ordered abelian group  $(C, +)$  and suppose  $A < |b|$  for all  $b \in B$ ,  $b \neq 0$ . Then  $A$  is a convex subgroup of  $A \oplus B$ .*

*Proof.* Clearly  $A + B$  is a direct sum. Let  $a, a' \in A$  and  $b \in B$  with  $0 < a + b < a'$ . From  $0 < a + b$  we get  $-b < a$ . From  $a + b < a'$  we get  $b < a' - a$ . Hence  $A \not< |b|$  and our assumption implies  $b = 0$ . ■

We return to  $\mathbf{k}((G))$  with prelogarithmic section  $l : G \rightarrow \mathbf{k}((G^{>1}))$ , prelogarithm  $L : (\mathbf{k}((G))^{>0}, \cdot) \rightarrow (\mathbf{k}((G)), +)$ , and associated EL-series field  $\mathbf{k}((G))^{EL}$  with logarithm  $\text{Log}$  and exponential  $\text{Exp}$ .

**Lemma 8.2.** *Let  $U \subseteq H \subseteq G$  be subgroups such that  $U$  is a proper convex subgroup of  $H$  ( $U = \{1\}$  is not excluded). Suppose*

$$(\dagger) \quad \text{Log}(H) < |f| \text{ for all } f \in \mathbf{k}((H^{>U})), f \neq 0.$$

Let

$$H^{\#,U} = H \cdot \text{Exp}(\mathbf{k}((H^{>U}))) \subseteq G^\#.$$

Then  $H$  is a proper convex subgroup of  $H^{\#,U}$ ,  $H^{\#,U}$  is the anti-lexicographic product of  $H$  and  $\text{Exp}(\mathbf{k}((H^{>U})))$  and

$$\text{Log}(H^{\#,U}) < |f| \text{ for all } f \in \mathbf{k}(((H^{\#,U})^{>H})), f \neq 0.$$

*Proof.*  $H$  is a proper subgroup of  $H^{\#,U}$ , since for every  $h \in H$  with  $h > U$  we know  $H < \text{Exp}(h)$  from  $(\dagger)$ . Let  $h^\# \in H^{\#,U}$ , hence  $h^\# = h \cdot \text{Exp}(f_1)$  with  $h \in H$  and  $f_1 \in \mathbf{k}((H^{>U}))$ . By assumption there is  $h_1 \in H$  with  $U < h_1$ . Then  $\text{Log}(h^\#) = \text{Log}(h) + f_1 < h_1 + f_1 \in \mathbf{k}((H))$ , hence for every  $f \in \mathbf{k}(((H^{\#,U})^{>H}))$  with  $f > 0$  we have  $\text{Log}(h^\#) \leq h_1 + f_1 < f$ . Finally Lemma 8.1 tells us that condition  $(\dagger)$  implies that  $H^{\#,U}$  is the anti-lexicographic product of  $H$  and  $\text{Exp}(\mathbf{k}((H^{>U})))$ . ■

**Definition 8.3.** Let  $H$  be a subgroup of  $G$  with  $H \neq \{1\}$ . We say that **the growth axiom holds for  $H$**  if condition  $(\dagger)$  of Lemma 8.2 is satisfied for  $H$  and  $U = \{1\}$ , in other words if

$$\text{Log}(H) < |f| \text{ for all } f \in \mathbf{k}((H^{>1})), f \neq 0.$$

Observe that for a simply generated subgroup  $H = h^{\mathbb{Z}}$  with  $h > 1$ ,  $H$  satisfies the growth axiom if and only if

$$v(\text{Log}(h)) < v(h)$$

where  $v$  denotes the natural valuation (in multiplicative notation), i.e. the valuation whose valuation ring is the convex hull of  $\mathbb{Z}$ . This is what classically is called “the growth axiom for  $h$ ”.

We iterate 8.2 to obtain

**Proposition 8.4.** *Let  $(\mathbf{k}, \log)$  be a logarithmic field and let  $(\mathbf{k}((G)), l)$  be a field with prelogarithmic section. Let  $H \neq \{1\}$  be a subgroup of  $G$ , satisfying the growth axiom (cf. Definition 8.3). Let  $H^n$  be defined inductively by  $H^{-1} = \{1\}$ ,  $H^0 = H$  and*

$$H^{n+1} = H^n \cdot \text{Exp } \mathbf{k}([H^n]^{>H^{n-1}})) \subseteq G^{\#n+1}.$$

*Then  $H^n$  is a proper convex subgroup of  $H^{n+1}$ ,  $H^{n+1}$  is the anti-lexicographical product of  $H^n$  and  $\text{Exp } \mathbf{k}([H^n]^{>H^{n-1}}))$  and*

$$\text{Log}(H^{n+1}) < |f| \text{ for all } f \in \mathbf{k}([H^{n+1}]^{>H^n}), f \neq 0.$$

*Proof.* By induction on  $n$ . If  $n = 0$ , then we may apply 8.2 for  $\{1\} \subseteq H$ , since  $H \neq \{1\}$  satisfies the growth axiom. Notice that  $H^1 = H^{\#,\{1\}}$  in the notation of 8.2 and we get the assertion for  $n = 0$ .

For the induction step we assume the assertion for  $n$  and apply 8.2 to the subgroups  $H^n \subsetneq H^{n+1}$  of  $G^{\#n+1}$ . Since  $H^{n+2} = (H^{n+1})^{\#H^n}$  in the notation of 8.2, we get the assertion for  $n + 1$ .  $\blacksquare$

**Lemma 8.5.** *If  $U \subseteq H$  are subgroups of  $G$  satisfying condition  $(\dagger)$  of Lemma 8.2 and  $\psi : (\mathbf{k}((G)), l) \rightarrow (\mathbf{k}((G)), l)$  is a morphism, then also  $\psi(U), \psi(H)$  satisfy condition  $(\dagger)$ .*

*Proof.* Take  $h_1 \in \psi(H)$  and let  $f_1 \in \mathbf{k}((\psi(H))^{>\psi(U)}), f_1 > 0$ . Then there are  $h \in H$  and  $f \in \mathbf{k}((H)^{>U}), f > 0$  with  $h_1 = \psi(h)$  and  $f_1 = \psi(f)$ . By assumption we have  $l(h) = \text{Log}(h) < f$ . Therefore  $\text{Log}(h_1) = l(\psi(h)) = \psi(l(h)) \leq \psi(f) = f_1$  as desired.  $\blacksquare$

## 9. LE-SERIES CONSTRUCTIONS.

Let  $(\mathbf{k}, \log)$  be a logarithmic field. Let  $(\mathbf{k}((G)), l)$  be a field with prelogarithmic section, and let  $\psi$  be a morphism from  $(\mathbf{k}((G)), l)$  to  $(\mathbf{k}((G)), l)$ , hence  $\psi$  is an order preserving embedding  $G \rightarrow G$  such that  $\psi \circ l = l \circ \psi$ . We fix a subgroup  $H \neq \{1\}$  of  $G$ . The goal of this section is to generalize the construction of the LE series field given in [DMM2, p. 67-72]. More precisely, we construct a subfield of  $\mathbf{k}((G))^{EL}$  which contains  $\mathbf{k}((H))$  and which is closed under  $\text{Exp}$ ,  $\text{Log}$  and  $\psi^{EL}$ , assuming natural conditions on  $H$  and  $\psi^{EL}$ . We shall call this field  $LE = LE(H, \psi)$  and suppress the dependency on  $H$  and  $\psi$  whenever the data  $H$  and  $\psi$  are clear from the context (cf. Theorem 9.6).

If  $m \in \mathbb{N}_0$  we write  $\psi^{(m)}$  for  $\overbrace{\psi \circ \dots \circ \psi}^{m \text{ times}}$ . Observe that  $\psi^{(m)}$  is again a morphism from  $(\mathbf{k}((G)), l)$  to  $(\mathbf{k}((G)), l)$ .

**Definition 9.1.** We define an increasing sequence of subgroups of  $G^{EL}$ ;

$$G_m^0 \subseteq G_m^1 \subseteq \dots \subseteq G_m^n \subseteq G_m^{n+1} \dots$$

with  $G_m^n \subseteq G^{\#n}$  as follows:

Let  $G_m^{-1} := \{1\}$ ,  $G_m^0 := \psi^{(m)}(H)$  which is an ordered multiplicative subgroup of  $G$ .

We define by induction on  $n$ :

$$G_m^{n+1} = G_m^n \cdot \text{Exp } \mathbf{k}([G_m^n]^{>G_m^{n-1}})) \subseteq G^{\#n+1}.$$

Set  $\mathbf{k}_m^n := \mathbf{k}([G_m^n])$  (so  $\mathbf{k}_m^n \subseteq \mathbf{k}_m^{n+1}$ ) and  $L_m := \bigcup_n \mathbf{k}_m^n$ .

We now study closure properties of the fields  $L_m$  under the various maps.

**Proposition 9.2.** *We have:*

- (1)  $\text{Exp}(\mathbf{k}_m^n) \subseteq \mathbf{k}_m^n \cdot G_m^{n+1} \subseteq \mathbf{k}_m^{n+1}$  (so  $L_m$  is closed under  $\text{Exp}$ ).
- (2)  $\psi^{EL}(G_m^n) = G_{m+1}^n$ .

*Proof.* (1). We have  $\mathbf{k}_m^n = \mathbf{k}([G_m^n]^{<G_m^{n-1}}) \oplus \mathbf{k}([G_m^n]^{=G_m^{n-1}}) \oplus \mathbf{k}([G_m^n]^{>G_m^{n-1}})$ . Since  $\text{Exp} \mathbf{k}([G_m^n]^{<G_m^{n-1}}) \subseteq \mathbf{k}([G_m^n]^{<G_m^{n-1}})$  we obtain

$$\text{Exp} \mathbf{k}_m^n \subseteq \mathbf{k}_m^n \cdot \text{Exp} \mathbf{k}([G_m^n]^{=G_m^{n-1}}) \cdot \text{Exp} \mathbf{k}([G_m^n]^{>G_m^{n-1}}).$$

By induction hypothesis we know  $\text{Exp} \mathbf{k}([G_m^n]^{=G_m^{n-1}}) \subseteq \mathbf{k}_m^n$  and by definition of  $G_m^{n+1}$  we have  $\text{Exp} \mathbf{k}([G_m^n]^{>G_m^{n-1}}) \subseteq G_m^{n+1}$ , so (1) follows. (2). If  $n = 0$ , then  $\psi(G_m^0) = G_{m+1}^0$  by definition. By Proposition 7.3,  $\psi^{EL}$  is a  $\mathbf{k}$ -homomorphism which respects  $\text{Exp}$  and arbitrary sums; by induction on  $n$ , we get  $\psi^{EL}(G_m^n) = G_{m+1}^n$  from the definition of  $G_m^n$ . ■

**Remark 9.3.** Notice that it does not follow from the definitions that  $G_m^{n+1}$  is the antilexicographic product of  $G_m^n$  and  $\text{Exp} \mathbf{k}([G_m^n]^{>G_m^{n-1}})$ . Similarly, it does not follow from the definitions that  $L_m \subseteq L_{m+1}$ . However, this will be the case under additional assumptions on  $l$ ,  $\psi$  and  $H$  which we will introduce step by step in the next statements. These results will be needed in section 10.

**Proposition 9.4.** *If  $H$  satisfies the growth axiom (cf. Definition 8.3) then for every  $m \in \mathbb{N}_0$ ,  $G_m^0$  satisfies the growth axiom and  $G_m^{n+1}$  is the anti-lexicographical product of  $G_m^n$  and  $\text{Exp} \mathbf{k}([G_m^n]^{>G_m^{n-1}})$ . Moreover we have*

$$\text{Exp} \mathbf{k}([G_m^n]^{>1}) \subseteq G_m^{n+1}.$$

*Proof.*  $G_m^0$  satisfies the growth axiom by 8.5 applied to  $H, \{1\}$  and the morphism  $\psi^{(m)}$ . Now 8.4 applied to  $G_m^0$  shows that  $G_m^{n+1}$  is the anti-lexicographical product of  $G_m^n$  and  $\text{Exp} \mathbf{k}([G_m^n]^{>G_m^{n-1}})$ . In particular  $G_m^n$  is a convex subgroup of  $G_m^{n+1}$ . To see the moreover part, take  $f \in \mathbf{k}([G_m^n]^{>1})$ . Since  $G_m^n$  is a convex subgroup of  $G_m^{n+1}$  we can write  $f = f_0 + \dots + f_n$ , where  $\text{supp } f_i \subseteq (G_m^i)^{>G_m^{i-1}}$ . By definition of  $G_m^{i+1}$  we have  $\text{Exp } f_i \in G_m^{i+1} \subseteq G_m^{n+1}$ . Thus  $\text{Exp } f = \prod_{i=0}^n \text{Exp } f_i \in G_m^{n+1}$ . ■

**Proposition 9.5.** *Assume  $H$  satisfies the growth axiom,  $\psi^{EL}$  is surjective and*

$$(17) \quad (\psi^{EL})^{-1}(G_0^0) \subseteq G_0^1 \text{ (i.e. } (\psi^{EL})^{-1}(H) \subseteq H \cdot \text{Exp} \mathbf{k}([H]^{>1})).$$

*Then for all  $n, m$  we have:*

- (1)  $(\psi^{EL})^{-1}(G_m^n) \subseteq G_m^{n+1}$ .
- (2)  $\mathbf{k}_m^n \subseteq \mathbf{k}_{m+1}^{n+1}$ .

*If in addition*

$$(18) \quad l(H) \subseteq \mathbf{k}(\psi(H)),$$

*then*

- (3)  $\text{Log } G_m^n \subseteq \mathbf{k}_{m+1}^n$  and

(4)  $\text{Log}(\mathbf{k}_m^n)^{>0} \subseteq \mathbf{k}_{m+1}^{n+1}$  for all  $n, m$ .

*Proof.* (1). We fix  $m$  and show (1) by induction on  $n$ . Firstly, since  $G_m^0 = \psi^{(m)}(H)$ , we have  $(\psi^{EL})^{-1}(G_m^0) = \psi^{(m)}(\psi^{-1}(H)) \subseteq \psi^{(m)}(G_0^1) = G_m^1$ , by assumption (17). Assume we know (1) for  $n$ . Then

$$(\psi^{EL})^{-1}(G_m^{n+1}) = (\psi^{EL})^{-1}(G_m^n) \cdot (\psi^{EL})^{-1}(\text{Exp } \mathbf{k}([G_m^n]^{>G_m^{n-1}})))$$

$\subseteq G_m^{n+1} \cdot \text{Exp } \mathbf{k}((\psi^{EL})^{-1}([G_m^n]^{>G_m^{n-1}})))$  by induction, so by definition of  $G_m^{n+2}$  it remains to show that  $\text{Exp } \mathbf{k}((\psi^{EL})^{-1}([G_m^n]^{>G_m^{n-1}}))) \subseteq G_m^{n+2}$ . By induction we know  $(\psi^{EL})^{-1}([G_m^n]^{>G_m^{n-1}})) \subseteq (G_m^{n+1})^{>1}$  and by 9.4 we know  $\text{Exp } \mathbf{k}((G_m^{n+1})^{>1})) \subseteq G_m^{n+2}$ . Hence the claim follows.

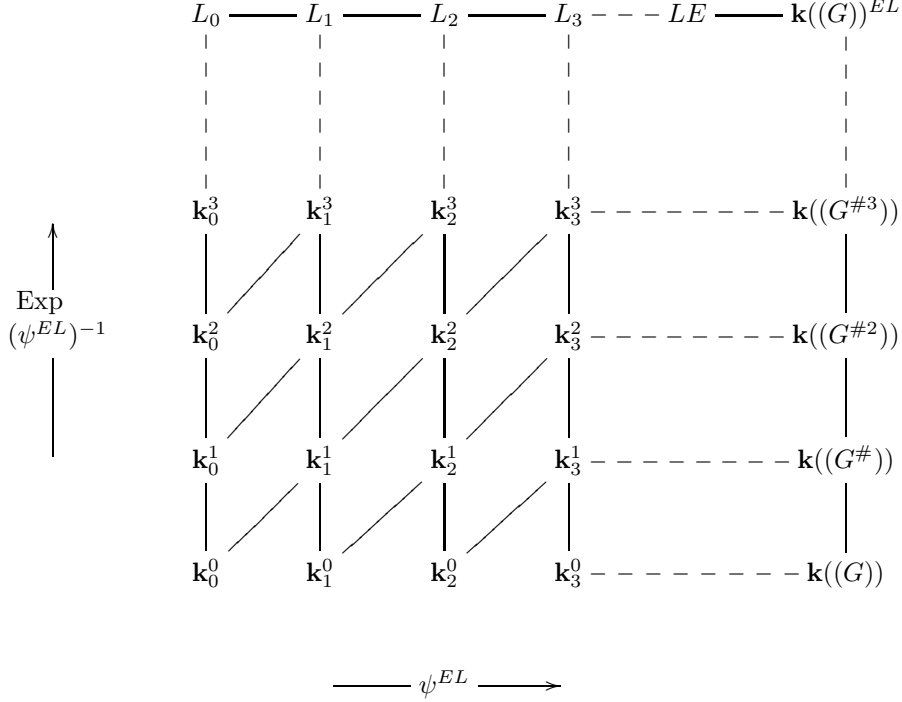
(2). By Proposition 9.2(2) we know  $G_m^n = (\psi^{EL})^{-1}(G_{m+1}^n)$ . Hence  $\mathbf{k}_m^n = \mathbf{k}((G_m^n)) = \mathbf{k}((\psi^{EL})^{-1}(G_{m+1}^n))$  and the latter field is contained in  $\mathbf{k}_{m+1}^{n+1} = \mathbf{k}((G_{m+1}^{n+1}))$  by (1).

Now assume that in addition  $l(H) \subseteq \mathbf{k}_1^1$ .

(3). We have  $\text{Log } G_m^0 = \text{Log } \psi^{(m)}(H) = \psi^{(m)}(\text{Log } H) \subseteq \psi^{(m)}(\mathbf{k}_1^0)$  by assumption. Since  $\psi^{(m)}(\mathbf{k}_1^0) = \mathbf{k}_{m+1}^0$  we get  $\text{Log } G_m^0 \subseteq \mathbf{k}_{m+1}^0$ . By definition of  $G_m^{n+1}$  we have  $\text{Log}(G_m^{n+1}) \subseteq \text{Log } G_m^n + \mathbf{k}_m^n$ . By (2) we know  $\mathbf{k}_m^n \subseteq \mathbf{k}_{m+1}^{n+1}$  and by induction on  $n$  we have  $\text{Log } G_m^n \subseteq \mathbf{k}_{m+1}^n$ . Since  $\mathbf{k}_{m+1}^n \subseteq \mathbf{k}_{m+1}^{n+1}$  we obtain (3).

(4). Since  $(\mathbf{k}_m^n)^{>0} = G_m^n \cdot \mathbf{k}^{>0} \cdot (1 + \mathbf{k}(((G_m^n)^{<1})))$  we get  $\text{Log}(\mathbf{k}_m^n)^{>0} \subseteq \text{Log}(G_m^n) + \mathbf{k}((G_m^n))$ . Now (2) says  $\mathbf{k}((G_m^n)) \subseteq \mathbf{k}_{m+1}^{n+1}$  and (3) says  $\text{Log}(G_m^n) \subseteq \mathbf{k}_{m+1}^n$ . Since  $\mathbf{k}_{m+1}^n \subseteq \mathbf{k}_{m+1}^{n+1}$  we obtain (4).  $\blacksquare$

Hence under the assumptions of Proposition 9.5, we have  $L_0 \subseteq L_1 \subseteq \dots$  and we define  $LE := LE(H; \psi) := \bigcup L_m$ . Here the situation in a diagram where a line indicates containment:



So far we know that  $LE$  is closed under  $\text{Exp}$ ,  $\text{Log}$  and  $\psi^{EL}$ .

**Theorem 9.6.** *Under the assumptions of Proposition 9.5, we have  $L_0 \subseteq L_1 \subseteq \dots$  and we define  $LE := LE(H; \psi) := \bigcup L_m$  - thus  $LE$  is closed under  $\text{Exp}$ ,  $\text{Log}$  and  $\psi^{EL}$ .*

We will now examine a sufficient condition on  $l$ ,  $\psi$  and  $H$  for assumption (17) of Proposition 9.5 to be fulfilled:

$$(\text{Comp}_H). \quad \psi|_{l^{-1}(G) \cap H} = l|_{l^{-1}(G) \cap H}$$

Notice that  $l^{-1}(G) = l^{-1}(G^{>1})$  as  $l$  has values in  $\mathbf{k}((G^{>1}))$ .

**Lemma 9.7.** *Let  $l$ ,  $\psi$ ,  $H$  be such that  $(\text{Comp}_H)$  holds. Then for every  $h \in l^{-1}(G) \cap H$ , and for every  $k \in \mathbf{k}$  we have:*

- (1)  $\psi^{EL}(\text{Exp}(k \cdot h)) = \text{Exp}(k \cdot \text{Log}(h))$ .
- (2) If  $\psi^{EL}$  is surjective, then  $(\psi^{EL})^{-1}(\text{Exp}(k \cdot \text{Log}(h))) \in G_0^1$ .

*Proof.* (1). Take  $h \in l^{-1}(G) \cap H$ . Then  $\psi^{EL}(\text{Exp}(k \cdot h)) = \text{Exp}(\psi^{EL}(k \cdot h)) = \text{Exp}(k \cdot \psi^{EL}(h)) = \text{Exp}(k \cdot \psi(h)) = \text{Exp}(k \cdot \text{Log}(h))$ . (2). By (1),  $(\psi^{EL})^{-1}(\text{Exp}(k \cdot \text{Log}(h))) = \text{Exp}(k \cdot h)$ . Since  $h \in l^{-1}(G) \cap H$  we have  $h > 1$ . Thus  $\text{Exp}(k \cdot h) \in G_0^1$  by definition.  $\blacksquare$

**Corollary 9.8.** *Let  $l$ ,  $\psi$ ,  $H_0$  be such that  $(\text{Comp}_{H_0})$  holds, and assume that  $\psi^{EL}$  is surjective. Let  $H$  be a subgroup of  $G$  containing  $H_0$  and contained in the group*



generated by all the  $\text{Exp}(k \cdot \text{Log}(h))$ ,  $k \in \mathbf{k}$ ,  $h \in l^{-1}(G) \cap H_0$ . Then assumption (17) of Proposition 9.5 is satisfied for  $H$ .

*Proof.* We must show that  $(\psi^{EL})^{-1}$  maps  $H$  into  $G_0^1 = H \cdot \text{Exp}(\mathbf{k}((H^{>1})))$ . By assumption on  $H$  it suffices to show that  $(\psi^{EL})^{-1}$  maps each element  $\text{Exp}(k \cdot \text{Log}(h))$ ,  $k \in \mathbf{k}$ ,  $h \in l^{-1}(G) \cap H_0$  into  $G_0^1$ . As  $(\text{Comp}_{H_0})$  is satisfied we may apply 9.7(2) for  $H_0$ , which precisely says this. ■

#### 10. FINDING THE LE-SERIES FIELD IN THE EXPONENTIAL FIELD GENERATED BY LOGARITHMIC WORDS.

Let  $G$  be the multiplicative group of logarithmic words:

$$G := \{x^{r_0} \cdot (\log x)^{r_1} \cdot \dots \cdot (\log_n x)^{r_n}, n \in \mathbb{N}_0, r_i \in k\}.$$

Define

$$\psi(x^{r_0} \cdot (\log x)^{r_1} \cdot \dots \cdot (\log_n x)^{r_n}) := (\log x)^{r_0} \cdot (\log_2 x)^{r_1} \cdot \dots \cdot (\log_{n+1} x)^{r_n}.$$

So  $\psi : G \longrightarrow G$  is an order preserving group embedding. Let  $l : G \longrightarrow \mathbf{k}((G^{>1}))$  be defined by

$$l(x^{r_0} \cdot (\log x)^{r_1} \cdot \dots \cdot (\log_n x)^{r_n}) := r_0 \cdot \log x + \dots + r_n \cdot \log_{n+1} x.$$

Then

- (1)  $l$  is a prelogarithmic section of  $\mathbf{k}((G))$  and  $\psi$  is a morphism from  $(\mathbf{k}((G)), l)$  to  $(\mathbf{k}((G)), l)$ . This is obvious.
- (2)  $\psi^{EL}$  is surjective, by Proposition 7.3(3) (applied to  $n = 1$ ).
- (3) Clearly  $\psi$  is a contracting morphism. Hence by Proposition 7.5,  $\psi^{EL}$  is a contraction. In particular  $(\psi^{EL})^{-1}(g) > g$  for all  $g \in G^{EL}$ ,  $g > 1$ .
- (4)  $H := \{x^k \mid k \in \mathbf{k}\}$  is a subgroup of  $G$  and  $H_0 := \{x^z \mid z \in \mathbb{Z}\}$  is a subgroup of  $H$ , generated by  $l^{-1}(G) \cap H_0 = l^{-1}(G) \cap H = \{x\}$ .
- (5)  $H$  satisfies the growth axiom (cf. Definition 8.3), since  $l(x^k) = k \log x < |f|$  for all  $f \in \mathbf{k}((H^{>1}))$ ,  $k \in \mathbf{k}$ .
- (6)  $\psi$  satisfies  $\text{Comp}_H$ , since  $l^{-1}(G) \cap H = \{x\}$  and  $\psi(x) = l(x)$ .
- (7)  $l$  and  $\psi$  satisfy condition (18) (cf. 9.5) as  $l(x^k) = k \log x = k\psi(x) \in \mathbf{k}((\psi(H)))$ .

Conditions (2),(3) and (6) show that all assumptions of 9.8 are satisfied. By this property (17) of 9.5 and by (7) also property (18) of 9.5 is satisfied. Hence the LE-series construction from section 9 is applicable and we show that it gives the field  $\mathbf{k}((x^{-1}))^{LE}$  of LE-series constructed by [DMM2, p. 67-72], where the composition with  $\text{Exp}(x)$  is

$$\Phi := (\psi^{EL})^{-1}.$$

By induction on  $n$  Proposition 9.4 identifies the group  $G_0^n$  with the group  $G_n$  from the construction in [DMM2]. Notice that by induction on  $n$  we may use our exponential function from  $\mathbf{k}((G))^{EL}$  as the abstract isomorphism taken in [DMM2] to define their exponential in step  $n$ . We see that  $L_0$  is the field  $\mathbf{k}((x^{-1}))^E$  from [DMM2]. Moreover we have shown that  $\Phi$  satisfies the following properties:

- (1)  $\Phi$  is an order preserving  $\mathbf{k}$ -isomorphism of exponential ordered fields.
- (2)  $\Phi(x^k) = \text{Exp}(k \cdot x)$  ( $k \in \mathbf{k}$ )
- (3)  $\Phi$  respects arbitrary sums.
- (4)  $L_m \subseteq L_{m+1}$
- (5)  $\Phi|_{L_{m+1}} : L_{m+1} \longrightarrow L_m$  is an isomorphism.

By induction on  $m$ , these properties identify the exponential fields  $L_m$  with the fields  $L_m$  from [DMM2] and the composition with  $\text{Exp}(x)$  (defined on  $L_m$ ) as  $(\psi^{EL})^{-1}$  (restricted to  $L_m$ ). This shows that the exponential field  $\mathbf{k}((x^{-1}))^{LE}$  is our exponential field  $LE$  and the composition with  $\text{Exp}(x)$  is  $(\psi^{EL})^{-1}$ .

## 11. THE $\mathbf{L}$ -OPERATION.

Let  $\Gamma$  be a totally ordered set,  $\sigma : \Gamma \longrightarrow \Gamma$  a decreasing embedding,  $(\mathbf{k}, \log)$  an ordered field with a surjective logarithm.

In this section, we show that the exponential-logarithmic field  $\mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma EL}$  (cf. Examples 4.5 and 6.4 and [KS]) does not embed in the logarithmic-exponential field  $\mathbf{k}((x^{-1}))^{LE}$  (cf. [DMM2]). We need the following notions:

**Definition 11.1.** Let  $(K_1, v_1)$  and  $(K_2, v_2)$  be valued fields with value groups  $G_1$  and  $G_2$  respectively. An embedding  $\phi : K_1 \longrightarrow K_2$  is said to be *valuation preserving* if the induced map  $v_1(a) \mapsto v_2(\phi(a))$  is a well defined order preserving embedding of  $G_1$  into  $G_2$ .

**Definition 11.2.** As in [DMM2, 6.27] we introduce the following notion. Fix  $\alpha \in \mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma EL}$  such that  $\alpha > k$  for all  $k \in \mathbf{k}$ . Let  $z \in \mathbb{Z}$ . Let  $s \in \mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma EL}$  such that  $s > k$  for all  $k \in \mathbf{k}$ . We say that  $s$  has *level*  $z$  with respect to  $\alpha$  if there exists  $N \in N_0$  such that  $v(\text{Log}_{N+z}(s)) = v(\text{Log}_N(\alpha))$ .

**Definition 11.3.** As in [DMM2], the exponential on  $\mathbf{k}((x^{-1}))^{LE}$  will be denoted by  $E$  and the logarithm by  $L$ . An embedding  $\varphi : \mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma EL} \longrightarrow \mathbf{k}((x^{-1}))^{LE}$  is said to be *exponential preserving* if it commutes with the exponential maps defined on its domain and co-domain, i. e. if  $E(\varphi(\alpha)) = \varphi(\text{Exp}(\alpha))$  for all  $\alpha \in \mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma EL}$ . Note that in this case,  $\varphi$  will also be *logarithm preserving*, that is,  $L(\varphi(\alpha)) = \varphi(\text{Log}(\alpha))$  for all  $\alpha \in \mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma EL}$ ,  $\alpha > 0$ .

**Theorem 11.4.** *Then there is no exponential  $\mathbf{k}$ -embedding of ordered fields from  $\mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma EL}$  into  $\mathbf{k}((x^{-1}))^{LE}$ .*

*Proof.* Fix an element  $X := x_\gamma$  for some  $\gamma \in \Gamma$  and consider

$$\mathbf{L} := \sum_{n \geq 0} \text{Log}_{\sigma, n} X.$$

Note that by definition,  $\text{Log}_{\sigma, n} X = \sigma_n(X)$  and the support of  $\mathbf{L}$  is anti wellordered, so  $\mathbf{L}$  is a well defined element of  $\mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma EL}$ .

Now suppose that there is an exponential  $\mathbf{k}$ -embedding  $\varphi : \mathbf{k}((\Gamma^{\mathbf{k}}))^{\sigma EL} \rightarrow \mathbf{k}((x^{-1}))^{LE}$ . Note that  $\varphi$  is valuation preserving since it is a  $\mathbf{k}$ -embedding. Let  $Y := \varphi(X) \in \mathbf{k}((x^{-1}))^{LE}$ . Then  $Y > \mathbf{k}$  and by [DMM2, Prop. 6.3] there is a compositional

inverse  $Y^{-1}$  of  $Y$  in  $\mathbf{k}((x^{-1}))^{LE}$ . By [DMM2, Th. 6.2] that means there is an  $E$ -preserving  $\mathbf{k}$ -embedding  $\psi : \mathbf{k}((x^{-1}))^{LE} \rightarrow \mathbf{k}((x^{-1}))^{LE}$  of ordered fields with  $\psi(Y) = x$ . So, by replacing  $\varphi$  with  $\psi \circ \varphi$  we may assume that  $\varphi(X) = x$ . By [DMM2, Lemma 6.21], there is some  $i \in \mathbb{N}$  such that  $\Phi^i(\varphi(\mathbf{L})) \in L_0$ . As  $\Phi^i : \mathbf{k}((x^{-1}))^{LE} \rightarrow \mathbf{k}((x^{-1}))^{LE}$  is an exp-preserving  $\mathbf{k}$ -embedding of ordered fields, we may replace  $\varphi$  by  $\Phi^i \circ \varphi$  and assume that  $\varphi(X) = E_i(x)$  and  $\varphi(\mathbf{L}) \in L_0$ . Let  $\mathbf{T} := \mathbf{L} - \sum_{n=0}^{i-1} \text{Log}_{\sigma,n} X$  and  $\mathbf{S} := \text{Log}_{\sigma,i} X$ . Then  $\varphi(\mathbf{S}) = x$  and  $\varphi(\mathbf{T}) \in L_0$ . Now  $\varphi$  is level preserving since it is both log-preserving and valuation preserving. Consider  $\mathbf{T} - \mathbf{S} > \mathbf{k}$ . We compute that  $\text{level}(\mathbf{T} - \mathbf{S}) < \text{level}(\mathbf{S})$  (w.r.t.  $\mathbf{S}$ ). Hence  $\text{level}(\varphi(\mathbf{T}) - x) < \text{level}(x) = 0$  (w.r.t.  $\varphi(\mathbf{S}) = x$ ). So  $\varphi(\mathbf{T}) - x \in L_0$  is an element  $> \mathbf{k}$  having a level  $< 0$ . This contradicts [DMM2, Prop. 6.28 p. 100]. ■

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